The Analytical Solutions for Magnetohydrodynamic Flow of a Third Order Fluid in a Porous Medium

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An analysis has been carried out for flow and heat transfer characteristics in a third grade fluid between two porous plates. The electrically conducting fluid fills the porous medium. The solutions have been developed for small porosity and magnetic field. Three flow problems are investigated and analytical expressions for the velocity field and temperature distribution are given for each case. Moreover, we recover and extend the results of Siddiqui et al. [1] by presenting exact solutions for the governing equations derived in [1].

Key words: Exact Analytical Solution; Third Grade Fluid; Plane Couette Flow; Plane Poiseuille Flow and Plane Couette-Poiseuille Flow; Magnetohydrodynamic Analysis and Porous Medium.

1. Introduction

The flows of non-Newtonian fluids are encountered in various industrial and technological applications. Various constitutive equations that can describe such fluids have been proposed. Several researchers are now engaged in studying the flow problems of these fluids. Some recent attempts may be mentioned in the references [2 – 11]. In Siddiqui et al. [1], the authors studied the heat transfer of a hydrodynamic third order fluid between two parallel plates which are kept at different temperatures. They looked at constant viscosity and treated three problems, via the plane Couette flow, plane Poiseuille flow and plane Couette-Poiseuille flow. For the hydrodynamic fluid and non-porous medium, the governing differential equations for velocity and temperature were derived in [1]. The derived non-linear equation has been solved by the homotopy perturbation method.

Biomagnetic fluid dynamics is currently a new area of research. It has several applications in bioengineering and medical sciences. Amongst these is the development of magnetic devices for cell separation, targeted transport of drugs using magnetic particles as drug earriers etc. Blood is an example of a magnetohydrodynamic fluid. This is because of complex interaction of intercellular protein, cell membranes and hemoglobin. Furthermore blood flow (with shear rate below 100 s^{-1}) in the coronary arteries represents a mathematical model of MHD non-Newtonian fluid in a porous medium. Therefore, in this paper, we extend the analysis of [1] in two directions (i) to analyze the magnetohydrodynamic (MHD) flow and (ii) to consider a porous medium. An approximate solution of the resulting non-linear problem is developed by employing perturbation method [12]. The obtained solutions are valid for small magnetic and porosity parameters. Furthermore, we recover the equations derived by Siddiqui et al. [1] and obtain the exact solutions whereas in [1] only approximate solutions were presented.

In the next section, we formulate three problems for MHD flow in a porous medium. Section 3 contains the solution procedure. Section 4 includes the discussion of the obtained results. In the same section, the present solutions are compared with the existing solutions given in [1]. Concluding remarks are mentioned in Section 5.

2. Mathematical Formulation

The MHD flow in a porous medium is governed by the continuity equation of motion, energy equation,

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and the Maxwell equations in the form

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\rho \frac{\mathrm{D}\mathbf{V}}{\mathrm{D}t} = \mathbf{J} \times \mathbf{B} + \nabla \cdot \mathbf{T} + \mathbf{R}, \tag{2}$$

$$\rho c_p \frac{\mathrm{D}\theta}{\mathrm{D}t} = k\nabla^2\theta + \mathbf{T} \cdot \mathbf{L}, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_m \mathbf{J}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4}$$

where $\mathbf{V}$ is the velocity field, $\mathbf{L} = \mathrm{grad}\mathbf{V}$, ρ the constant fluid density, $\mathbf{J}$ the current density, $\mathbf{B}$ the total magnetic field, $\mathbf{E}$ the total electric field, μ_m the magnetic permeability, $\mathbf{R}$ the Darcy's resistance, $\mathbf{T}$ the stress tensor, c_p the specific heat, k the thermal conductivity, θ the temperature, and $\mathrm{D}/\mathrm{D}t$ the material derivative. Note that in writing the Maxwell's equation the displacement current is neglected. By Ohm's law the expression of $\mathbf{J}$ is

$$\mathbf{J} = \sigma[\mathbf{E} + \mathbf{V} \times \mathbf{B}], \tag{5}$$

where σ is the electrical conductivity.

The constitutive equation for a third order fluid is

$$\begin{aligned}\mathbf{T} = {} & -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_1\mathbf{A}_3 \\ & + \beta_2(\mathbf{A}_1\mathbf{A}_2 + \mathbf{A}_2\mathbf{A}_1) + \beta_3(\mathrm{tr}\mathbf{A}_1^2)\mathbf{A}_1,\end{aligned} \tag{6}$$

where p is the fluid pressure, μ the coefficient of viscosity, α_i $(i = 1,2)$, β_j $(j = 1-3)$ the material constants, and $\mathbf{A}_i$ $(i = 1-3)$ the Rivlin-Ericksen tensors

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \tag{7}$$

$$\mathbf{A}_n = \frac{\mathrm{D}\mathbf{A}_{n-1}}{\mathrm{D}t} + \mathbf{A}_{n-1}\mathbf{L} + \mathbf{L}^T\mathbf{A}_{n-1}, \quad n = 2,3. \tag{8}$$

We now consider the following three problems.

2.1. Plane Couette Flow

As in [1], we first investigate the steady flow of a third order fluid between two infinite parallel plates distant $2h$ apart. The upper and lower plates are at $y = h$ and $y = -h$ of a rectangular system with the x-axis as flow direction. The upper plate is assumed to be moving with constant speed U and the lower kept stationary. The temperature of the higher and the lower plates are θ_1 and θ_2, respectively. We consider MHD unidirectional flow with zero pressure gradient. A uniform applied magnetic field B_0 acts in the y-direction. The induced magnetic field is neglected under the assumption of small magnetic Reynolds number. No polarization effects are included and hence the electric field is chosen zero. Thus we have $\theta(y)$ and

$$\mathbf{V} = [u(y), 0, 0]. \tag{9}$$

The above definition of velocity satisfies the continuity equation. In porous space the momentum and energy equations yield

$$\begin{aligned}& \mu\frac{\mathrm{d}^2u}{\mathrm{d}y^2} + 6(\beta_2+\beta_3)\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\frac{\mathrm{d}^2u}{\mathrm{d}y^2} - \sigma B_0^2 u \\ & -\frac{\phi}{k_1}\left[\mu + 2(\beta_2+\beta_3)\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right]u = 0,\end{aligned} \tag{10}$$

$$k\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} + \mu\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 + 2(\beta_2+\beta_3)\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^4 = 0, \tag{11}$$

in which the applied magnetic field B_0 and porosity ϕ are constants. Here k_1 is the permeability of the porous medium. The appropriate conditions are [1]

$$\begin{aligned}& u(-h) = 0, \quad u(h) = U, \\ & \theta(-h) = \theta_0, \quad \theta(h) = \theta_1.\end{aligned} \tag{12}$$

Defining

$$u = Uu^*, \quad y = y^*h, \quad \theta^* = \frac{\theta - \theta_0}{\theta_1 - \theta_0}, \tag{13}$$

(10) and (11) become after omitting the asterisks

$$\begin{aligned}& \frac{\mathrm{d}^2u}{\mathrm{d}y^2} + 6\beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\frac{\mathrm{d}^2u}{\mathrm{d}y^2} - M^2u \\ & -\frac{1}{K}\left[1 + 2\beta\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2\right]u = 0,\end{aligned} \tag{14}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} + \lambda\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 + 2\beta\lambda\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^4 = 0, \tag{15}$$

with the boundary conditions

$$\begin{aligned}& u(-1) = 0, \quad u(1) = 1, \\ & \theta(-1) = 0, \quad \theta(1) = 1,\end{aligned} \tag{16}$$

where

$$\begin{aligned}& \beta = \frac{\beta_2+\beta_3}{\mu}\left(\frac{U}{h}\right)^2, \; M^2 = \frac{\sigma B_0^2}{\mu}h^2, \; \frac{1}{K} = \frac{\phi}{k_1}h^2, \\ & \lambda = \frac{\mu U^2}{k(\theta_1-\theta_0)} = \frac{\mu c_p}{k} \times \frac{U^2}{c_p(\theta_1-\theta_0)} = P_\mathrm{r}E_\mathrm{c},\end{aligned} \tag{17}$$

in which P_r and E_c are the respective Prandtl and Eckert numbers and P_rE_c the Brinkman number. In the limits $M \to 0$ and $K \to \infty$, (14) and (16) reduce to the problem studied in [1].

2.2. Plane Poiseuille Flow

As in [1], we now look at the situation when both the plates are stationary and the fluid motion is due to constant pressure gradient. In this case one directly obtains after non-dimensionalization:

$$\frac{d^2u}{dy^2} + 6\beta\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} - M^2u - \frac{1}{K}\left[1 + 2\beta\left(\frac{du}{dy}\right)^2\right]u = -B, \tag{18}$$

$$\frac{d^2\theta}{dy^2} + \lambda\left(\frac{du}{dy}\right)^2 + 2\beta\lambda\left(\frac{du}{dy}\right)^4 = 0, \tag{19}$$

where in addition to (17) we have

$$B = -\frac{h^2}{\mu U}\frac{d\hat{p}}{dx}, \quad \hat{p} = p - (2\alpha_1 + \alpha_2)\left(\frac{du}{dy}\right)^2 \tag{20}$$

with modified pressure $\hat{p} = \hat{p}(x)$ and hence B constant. The boundary conditions here are

$$u(-1) = 0, \quad u(1) = 0, \theta(-1) = 0, \quad \theta(1) = 1. \tag{21}$$

2.3. Plane Couette-Poiseuille Flow

The fluid motion in this final case is by motion of the upper plate with constant velocity U and pressure gradient [1]. After scaling, given by (13) and (20) one has (18), (19) subject to the boundary conditions

$$\begin{aligned} &u(-1) = 0, \quad u(1) = 1, \\ &\theta(-1) = 0, \quad \theta(1) = 1. \end{aligned} \tag{22}$$

In the next section we solve these three problems by the perturbation expansion approach. However, our base equation will be retained as non-linear compared to the linear ones that are normally used. This also enables us to derive the exact solutions for the governing equations of [1].

3. Solution Procedure of the Problem

We take

$$M^2 = N^2\varepsilon, \quad \frac{1}{K} = \varepsilon, \tag{23}$$

where ε is a small parameter, N^2 a constant, and M^2 is of order ε. We are therefore assuming weak magnetic field and weak porosity. Now let be

$$u = u_0 + \varepsilon u_1 + O(\varepsilon^2). \tag{24}$$

We first investigate solutions for section (2.1), i. e. plane Couette flow. Then (14) becomes for zeroth and first order in ε

$$u_0'' + 6\beta u_0'^2 u_0'' = 0, \tag{25}$$

$$\begin{aligned} &u_1'' + 12\beta u_0' u_0'' u_1' + 6\beta u_0'^2 u_1'' \\ &- u_0(N^2+1) - 2\beta u_0'^2 u_0 = 0, \end{aligned} \tag{26}$$

where primes denote the derivative with respect to y.

Equation (26) can be written as

$$\frac{d}{dy}(u_1'(1 + 6\beta u_0'^2)) = u_0(N^2+1) + 2\beta u_0'^2 u_0. \tag{27}$$

Equation (25) has first integral

$$u_0' + 2\beta u_0'^3 = D, \tag{28}$$

for D constant. One can solve for u_0' to arrive at

$$u_0' = \sqrt[3]{\frac{1}{4\beta}D + \frac{1}{4\beta}\sqrt{D^2 + \frac{2}{27\beta}}} - \sqrt[3]{-\frac{1}{4\beta}D + \frac{1}{4\beta}\sqrt{D^2 + \frac{2}{27\beta}}}, \tag{29}$$

which gives

$$u_0 = \Omega y + c_0, \tag{30}$$

where Ω is the right side of (29) and c_0 a constant of integration. The first set of boundary conditions of (16) results in

$$u_0 = \frac{1}{2}(y+1) \tag{31}$$

with $\Omega = 1/2$. The solution of (27) for u_1 subject to

$$u_1(-1) = 0, \quad u_1(1) = 0, \tag{32}$$

then yields

$$\begin{aligned} u_1 = &\frac{N^2+1}{12\left(1+\frac{3}{2}\beta\right)}y^3 + \frac{\beta}{24\left(1+\frac{3}{2}\beta\right)}y^3 \\ &+ \frac{N^2+1}{4\left(1+\frac{3}{2}\beta\right)}y^2 + \frac{\beta}{8\left(1+\frac{3}{2}\beta\right)}y^2 \\ &+ c_1 y + c_2, \end{aligned} \tag{33}$$

where c_1 and c_2 are

$$c_1 = -\frac{N^2+1}{12\left(1+\frac{3}{2}\beta\right)} - \frac{\beta}{12\left(1+\frac{3}{2}\beta\right)}, \quad c_2 = -\frac{N^2+1}{4\left(1+\frac{3}{2}\beta\right)} - \frac{\beta}{6\left(1+\frac{3}{2}\beta\right)}. \tag{34}$$

The θ can be obtained by direct integration subject to the boundary condition (16) (second set) and gives rise to

$$\theta = M_1 + M_2 y + M_3 y^2 + M_4 y^3 + M_5 y^4 + M_6 y^5 + M_7 y^6 + M_8 y^7 + M_9 y^8 + M_{10} y^9 + M_{11} y^{10}, \tag{35}$$

where the constants $M_1 - M_{11}$ can be calculated through simple computations.

Next we consider approximate solutions for the problem in section 2.2, via plane Poiseuille flow.

We first use the rescaling

$$u = Bu^*, \quad \beta^* = \beta B^2 \tag{36}$$

to scale B to unity. We again assume a solution of the form (24). Then the zeroth order equation is now (set $B = 1$ and write β instead of β^*)

$$u_0'' + 6\beta u_0'^2 u_0'' = -1. \tag{37}$$

A first integral of (37) is

$$u_0' + 2\beta u_0'^3 + y = D, \tag{38}$$

with D constant. Equation (38) can be solved for u_0 as follows:

$$u_0' = \sqrt[3]{\frac{1}{4\beta}(-y+D) + \frac{1}{4\beta}\sqrt{(-y+D)^2 + \frac{2}{27\beta}}} + \sqrt[3]{\frac{1}{4\beta}(-y+D) - \frac{1}{4\beta}\sqrt{(-y+D)^2 + \frac{2}{27\beta}}}. \tag{39}$$

Under the transformation

$$\bar{y} = -y + D, \tag{40}$$

equation (39) becomes

$$\frac{\mathrm{d}u_0}{\mathrm{d}\bar{y}} = -\sqrt[3]{\frac{1}{4\beta}\bar{y} + \frac{1}{4\beta}\sqrt{\bar{y}^2 + \frac{2}{27\beta}}} - \sqrt[3]{\frac{1}{4\beta}\bar{y} - \frac{1}{4\beta}\sqrt{\bar{y}^2 + \frac{2}{27\beta}}}. \tag{41}$$

The boundary conditions

$$u_0(-1) = 0, \quad u_0(1) = 0$$

become

$$u_0(1+D) = 0, \quad u_0(-1+D) = 0.$$

Since D appears in the boundary condition we can set $D = 0$ to get

$$u_0(1) = 0, \quad u_0(-1) = 0. \tag{42}$$

The solution of (39) and (42) is

$$\begin{aligned} u_0 = {} & \frac{-9y}{8(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81y^2+\frac{6}{\beta}}+9y}{\beta}} - \sqrt[3]{\frac{\sqrt{81y^2+\frac{6}{\beta}}-9y}{\beta}}\right] \\ & + \frac{1}{24(6)^{\frac{2}{3}}}\sqrt{81y^2+\frac{6}{\beta}}\left[\sqrt[3]{\frac{\sqrt{81y^2+\frac{6}{\beta}}-9y}{\beta}} + \sqrt[3]{\frac{\sqrt{81y^2+\frac{6}{\beta}}+9y}{\beta}}\right] \\ & + \frac{9}{8(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81+\frac{6}{\beta}}+9}{\beta}} - \sqrt[3]{\frac{\sqrt{81+\frac{6}{\beta}}-9}{\beta}}\right] - \frac{1}{24(6)^{\frac{2}{3}}}\sqrt{81+\frac{6}{\beta}}\left[\sqrt[3]{\frac{\sqrt{81+\frac{6}{\beta}}+9}{\beta}} + \sqrt[3]{\frac{\sqrt{81+\frac{6}{\beta}}-9}{\beta}}\right]. \end{aligned} \tag{43}$$

We use (43) in (26) subject to the boundary conditions

$$u_1(-1) = 0, \quad u_1(1) = 0. \tag{44}$$

Since (26) is integrable by quadrature we plot the solutions (24) for different values of the parameters using Mathematica. We also invoke (19) to find and plot the graphs of θ by setting $u = u_0 + \varepsilon u_1$. Note that in (19) β is now replaced by β/B^2 due to the rescaling (36). These calculations are not explicitly given but they can be verified easily by Mathematica.

For the problem in section 2.3 we have

$$u_0(-1) = 1, \quad u_0(1) = 0, \tag{45}$$

and therefore

$$\begin{aligned} u_0 = {} & \frac{9(1-y)}{8(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{81(1-y)^2+\frac{6}{\beta}}-9(1-y)}{\beta}}-\sqrt[3]{\frac{\sqrt{81(1-y)^2+\frac{6}{\beta}}+9(1-y)}{\beta}}\right] \\ & + \frac{1}{24(6)^{\frac{2}{3}}}\sqrt{81(1-y)^2+\frac{6}{\beta}}\left[\sqrt[3]{\frac{\sqrt{81(1-y)^2+\frac{6}{\beta}}-9(1-y)}{\beta}}+\sqrt[3]{\frac{\sqrt{81(1-y)^2+\frac{6}{\beta}}+9(1-y)}{\beta}}\right]-\frac{1}{12\beta}, \end{aligned} \tag{46}$$

where

$$\begin{aligned} & \frac{9}{4(6)^{\frac{2}{3}}}\left[\sqrt[3]{\frac{\sqrt{324+\frac{6}{\beta}}-18}{\beta}}-\sqrt[3]{\frac{\sqrt{324+\frac{6}{\beta}}+18}{\beta}}\right] \\ & \quad + \frac{1}{24(6)^{\frac{2}{3}}}\sqrt{324+\frac{6}{\beta}}\left[\sqrt[3]{\frac{\sqrt{324+\frac{6}{\beta}}-18}{\beta}}+\sqrt[3]{\frac{\sqrt{324+\frac{6}{\beta}}+18}{\beta}}\right]-\frac{1}{12\beta}=1. \end{aligned} \tag{47}$$

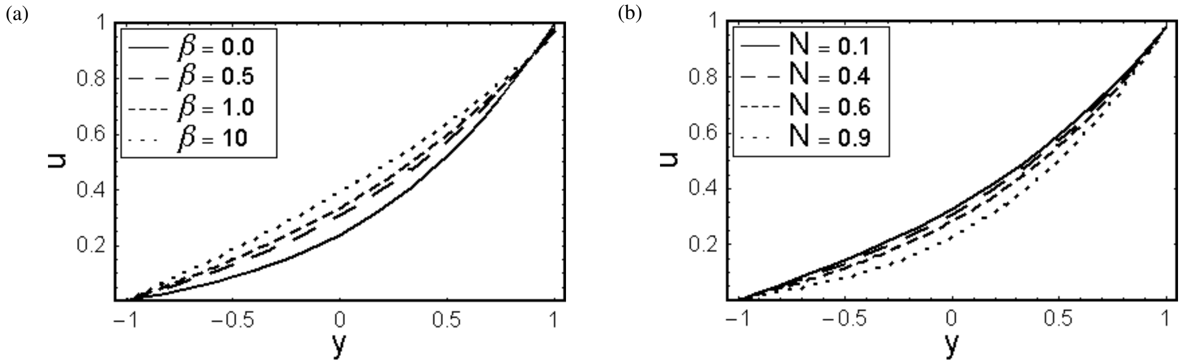
Then u_1 is obtained from (26) with the conditions

$$u_1(1) = 0, \quad u_1(-1) = 0. \tag{48}$$

In the same manner (26) is solvable by quadrature and we utilize Mathematica to plot solutions of (19) and (24). We do not present the solutions as they are lengthy.

4. Results and Discussion

In this section we assign the physical interpretations to velocity and temperature profiles. For that we first discuss the expressions of velocity and temperature profiles for plane Couette flow, plane Poiseuille flow and plane Couette-Poiseuille flow. For velocity profile of plane Couette flow, Figure 1 is plotted to examine the effect of third order parameter β, MHD parameter N, and porosity parameter ε. It is observed that the velocity increases with an increase in the value of β. The effects of N and ε on u are similar, that is the velocity decreases by increasing N and ε, respectively. In Figure 2, we present the effects of N, β, ε, and Brinkman number λ on the temperature profiles θ. It is seen that the boundary layer thickness decreases with the increase of N, β, ε and λ. Figures 3 – 6 show the variation of velocity and temperature profiles for plane Poiseuille flow and plane Couette-Poiseuille flow, respectively, when B is equal to one. Here Figures 3 and 4 are for plane Poiseuille flow whereas Figures 5 and 6 are for plane Couette-Poiseuille flow. We observe that the behavior of the velocity and temperature profiles in



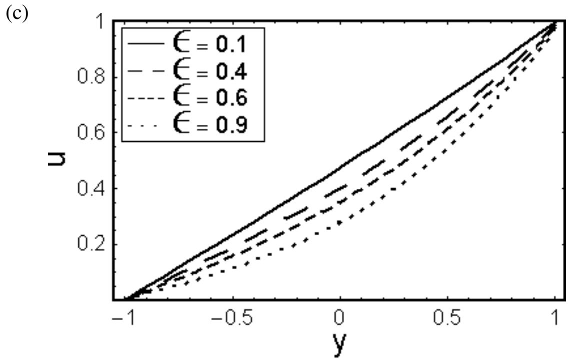


Fig. 1. Effect of the parameters β (a), N (b), ε (c), respectively, on the velocity field $u(y)$ for plane Couette flow when (a): $N = 0.5$ and $\varepsilon = 0.9$; (b): $\beta = 0.5$ and $\varepsilon = 0.9$; (c): $\beta = 0.5$ and $N = 0.9$.

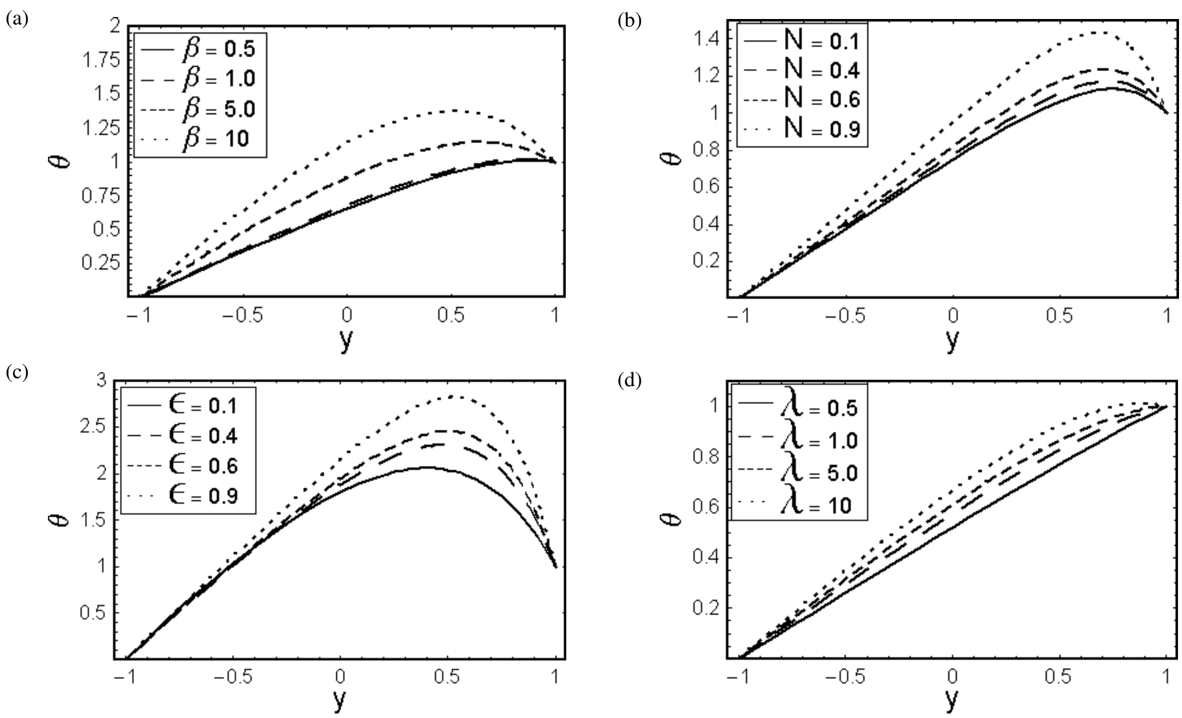
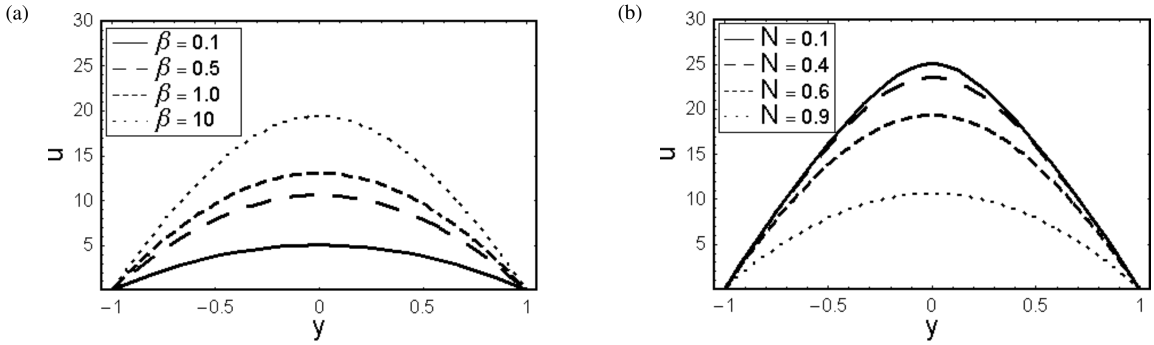


Fig. 2. Effect of the parameters β (a), N (b), ε (c) and λ (d), respectively, on the temperature field $\theta(y)$ for plane Couette flow when (a): $N = 0.5$, $\varepsilon = 0.9$ and $\lambda = 1$; (b): $\varepsilon = 0.9$ and $\lambda = \beta = 1$; (c); $N = 0.5$ and $\lambda = \beta = 1$; (d): $N = 0.5$, $\varepsilon = 0.9$ and $\beta = 1$.



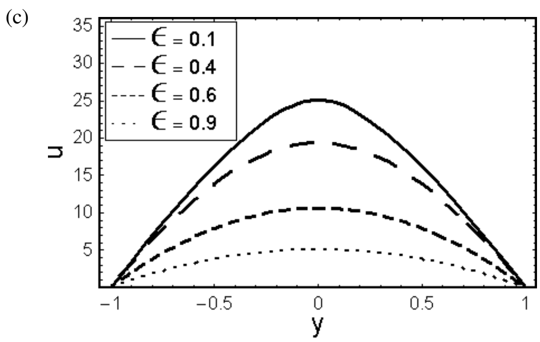


Fig. 3. Effect of the parameters β (a), N (b), and ε (c), respectively, on the velocity field $u(y)$ for plane Poiseuille flow when (a): $N = 0.5$ and $\varepsilon = 0.9$; (b): $\beta = 0.5$ and $\varepsilon = 0.9$; (c): $\beta = 0.5$ and $N = 0.9$.

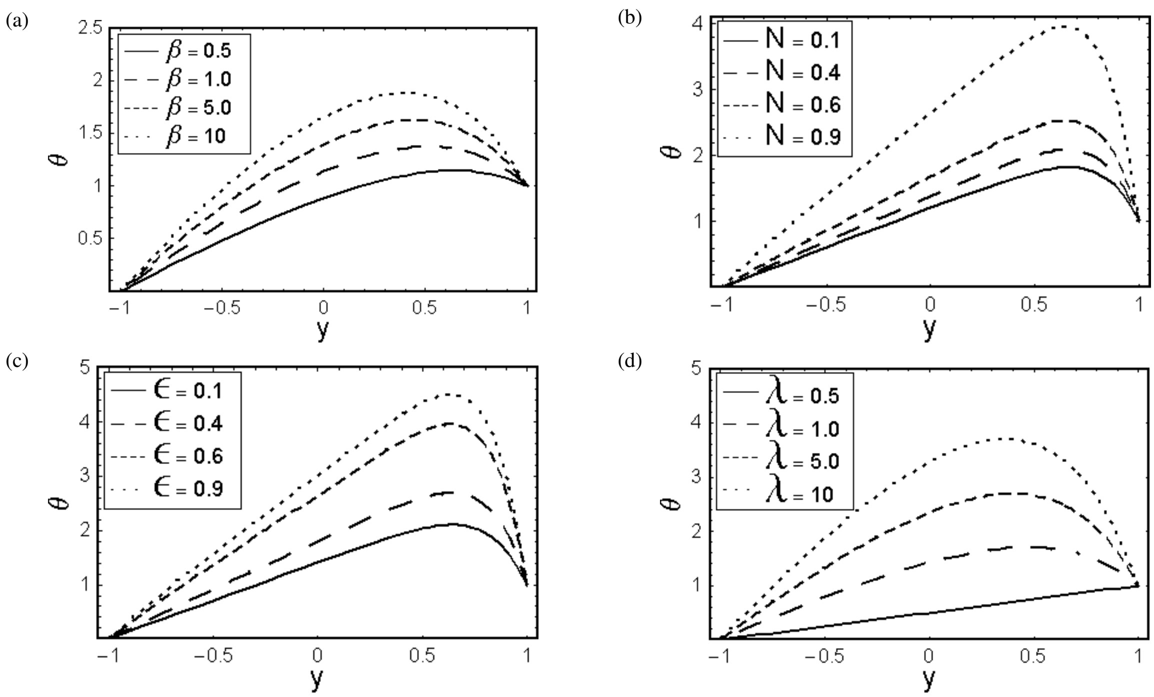
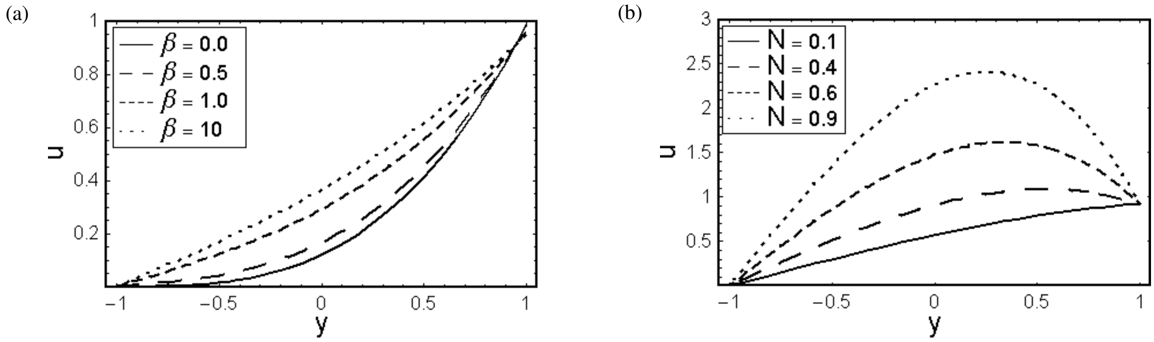


Fig. 4. Effect of the parameters β (a), N (b), ε (c), and λ (d), respectively, on the temperature field $\theta(y)$ for plane Poiseulle flow when (a): $N = 0.5$, $\varepsilon = 0.9$ and $\lambda = 1$; (b): $\varepsilon = 0.9$ and $\lambda = \beta = 1$; (c): $N = 0.9$ and $\lambda = \beta = 1$; (d): $N = 0.5$, $\varepsilon = 0.9$ and $\beta = 1$.



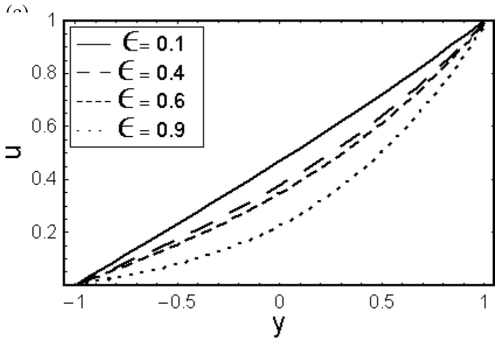


Fig. 5. Effect of the parameters β (a), N (b), and ε (c) on the velocity field $u(y)$ for plane Couette-Poiseuille flow when (a): $N = 0.5$ and $\varepsilon = 0.9$; (b): $\beta = 0.5$ and $\varepsilon = 0.9$; (c): $\beta = 0.5$ and $N = 0.9$.

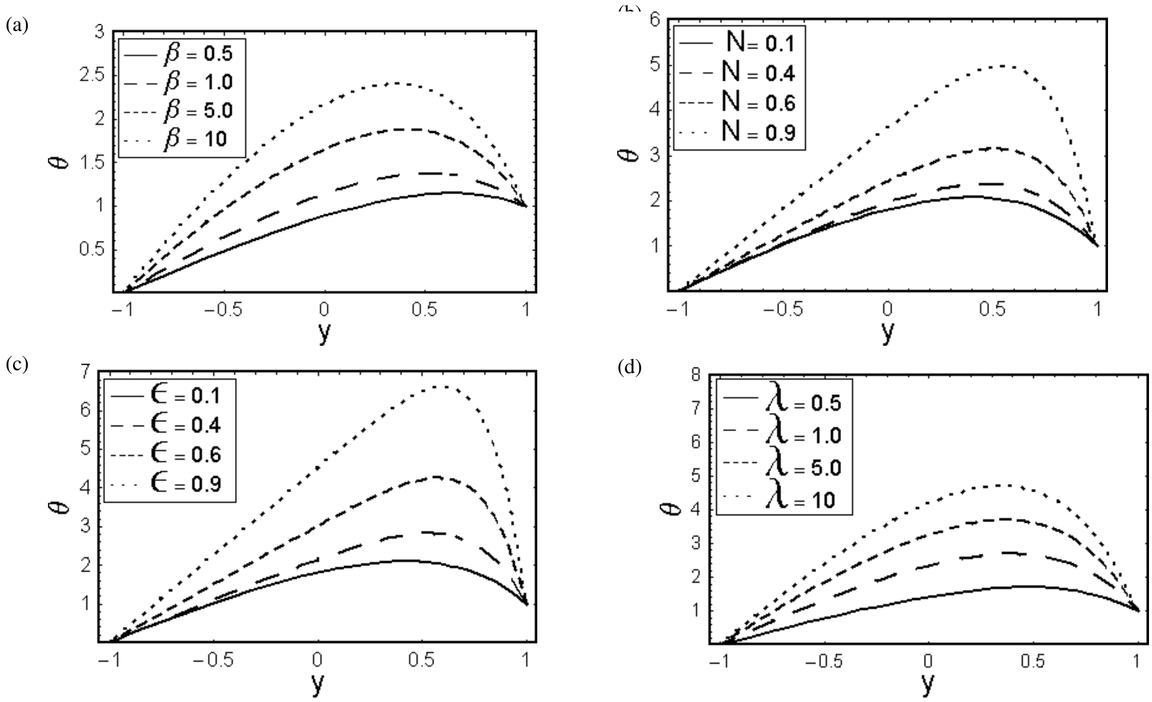


Fig. 6. Effect of the parameters β (a), N (b), ε (c), and λ (d) on the temperature field $\theta(y)$ for plane Couette-Poiseulle flow when (a): $N = 0.5$, $\varepsilon = 0.9$ and $\lambda = 1$; (b): $\varepsilon = 0.9$ and $\lambda = \beta = 1$; (c): $N = 0.9$ and $\lambda = \beta = 1$; (d): $N = 0.5$, $\varepsilon = 0.9$ and $\beta = 1$.

Table 1. Relation between Ω and D given in [27].

D	0	0.5	1.0	1.5	2.0	2.5	3.0
Ω	0	0.3411	0.4999	0.6067	0.6894	0.7580	0.8172

Table 2. Relation between β and D for $\Omega = 1/2$.

β	0	0.2	0.4	0.6	0.8	1.0	2.0	4.0
D	0	0.55	0.60	0.65	0.70	0.75	0.1	1.50

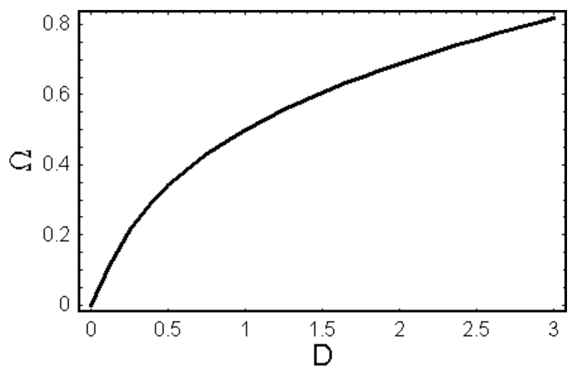


Fig. 7. The variation of Ω as a function of D given in the right hand side of (27).

both cases is the same as that of Figures 1 and 2, except for somewhat larger velocity and temperature rise in every later case. Figure 7 shows the variation of Ω as a function of D given in the right hand side of (29). The numerical relation of Ω as a function of D is also given in Table 1. The velocity condition at $\Omega = 1/2$ for various values of β satisfies the relation given in Table 2.

5. Concluding Remarks

In this paper, three flow problems namely plane Couette flow, plane Poiseuille flow, and plane Couette-Poiseuille flow of a third order fluid between two parallel heated plates with weak MHD and porosity effects are investigated. We note the following features:

• Our solutions (31), (43) and (46) are exact whereas in [1] only approximate solutions are presented.

• In order to compare our results with those of [1], we set $\varepsilon = 0$. Also

$$\beta = B^2 \beta^*, \tag{49}$$

where β^* is that of [1].

• In addition we have modified [1] and observe the effect of weak magnetic field and porosity on both velocity and temperature profiles and found that the boundary layer thickness decreases with the increasing parameters of N and ε.

• We can achieve the results of Newtonian fluid for plane Couette flow when $\beta = 0$.

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